Notes on *Notes on Constructive Set Theory*

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Abstract

These notes, which contain the content of a talk given by the author at the ILLC in Amsterdam on the above date, outline some selected topics in constructive set theory. Following [1], we present two axiomatizations of constructive set theory: **ECST** and **CZF**. For the first topic, we give two constructions of the reals and discuss when they are equivalent. We next move to well-founded relations, defining ordinals and proving some simple facts about induction. For the last topic, we analyze several choice principles and their interconnections, finishing with some principles that ought to be avoided.

Contents

1 Axioms

We first recall the axiom systems **ECST** and **CZF**, as presented in [1]. Both use intuitionistic logic.

1.1 ECST

Extensionality	$\forall a \forall b [\forall x (x \in a \leftrightarrow x \in b) \to a = b]$
Pairing	$\forall a \forall b \exists y \forall u [u \in y \leftrightarrow y = a \vee y = b]$
Union	$\forall a \exists y \forall x [x \in y \leftrightarrow \exists u \in a (w \in u)]$
Strong Infinity	$\exists a [\mathtt{Ind}(a) \wedge \forall b [\mathtt{Ind}(b) \to \forall x \in a (x \in b)]]$
Δ_0-Separation	$\forall a \exists y \forall x [x \in y \leftrightarrow x \in a \wedge \varphi(x)]$ for all $\varphi(x) \in \Delta_0$
Replacement	$\forall x \in a \exists ! y \varphi(x, y) \to \exists b \forall y [y \in b \leftrightarrow \exists x \in a \varphi(x, y)]$ for all formulas $\varphi(x, y)$ without b free

Note that Strong Infinity relies on the following abbreviations:

$$\begin{aligned} \mathtt{Empty}(y) &:\equiv (\forall z \in y) \bot \\ \mathtt{Succ}(x, y) &:\equiv \forall z [z \in y \leftrightarrow z \in x \vee z = x] \\ \mathtt{Ind}(a) &:\equiv (\exists y \in a) \mathtt{Empty}(y) \wedge (\forall x \in a)(\exists y \in a) \mathtt{Succ}(x, y) \end{aligned}$$

where $\equiv$ denotes syntactic equality.

1.2 CZF

Extensionality	$\forall a \forall b [\forall x (x \in a \leftrightarrow x \in b) \to a = b]$
Pairing	$\forall a \forall b \exists y \forall u [u \in y \leftrightarrow y = a \vee y = b]$
Union	$\forall a \exists y \forall x [x \in y \leftrightarrow \exists u \in a (w \in u)]$
Strong Infinity	$\exists a [\mathtt{Ind}(a) \wedge \forall b [\mathtt{Ind}(b) \to \forall x \in a (x \in b)]]$
Set Induction	$\forall a [\forall x \in a \varphi(x) \to \varphi(a)] \to \forall a \varphi(a)$
Δ_0-Separation	$\forall a \exists y \forall x [x \in y \leftrightarrow x \in a \wedge \varphi(x)]$ for all $\varphi(x) \in \Delta_0$
Strong Collectoin	$\forall x \in a \exists y \varphi(x, y) \to \exists b [\forall x \in a \exists y \in b \varphi(x, y) \wedge \forall y \in b \exists x \in a \varphi(x, y)]$ for all formulas $\varphi(x, y)$
Subset Collection	$\exists c \forall u [\forall x \in a \exists y \in b \psi(x, y, u) \to \exists d \in c (\forall x \in a \exists y \in d \psi(x, y, u) \wedge \forall y \in d \exists x \in a \psi(x, y, u))]$ for all formulas $\psi(x, y, u)$

2 Continuum

In this section, assuming that the natural numbers $\mathbb{N}$ and the rationals $\mathbb{Q}$ have been constructed, we present two constructions of the continuum. The first mimics the classical approach of Dedekind cuts, while the second uses Cauchy sequences for a more traditionally constructive flavor. The section finishes with some statements on the relation between these two notions of reals.

2.1 Dedekind reals

The classical Dedekind reals are obtained by taking proper, nonempty subsets X of rationals such that $r \in X \Leftrightarrow (\exists s \in X)[r < s]$: the so-called Dedekind cuts. To obtain the analogous structure in constructive set theory, though, an additional requirement is needed.

Definition 2.1.1. *A* cut *is a proper, nonempty $X \subset \mathbb{Q}$ such that*

$$X = X^{<} := \{r \in \mathbb{Q} : (\exists s \in X) r < s\}$$

and for all $r, r' \in \mathbb{Q}$,

$$r < r' \Rightarrow [r \in X \vee r' \notin X].$$

With this we can define our fist notion of the constructive reals.

Definition 2.1.2. *The* Constructive Dedekind reals $\mathbb{R}$ *is the class of all left cuts. We define the relation $<$ on $\mathbb{R}$ by $X < Y$ if there is a rational in $Y \backslash X$.*

2.2 Cauchy reals

The traditional approach to constructive reals defines them as equivalence classes of certain sequences of rationals. Here we taking a slightly different approach, instead relating the sequences to the previously defined Dedekind cuts.

Definition 2.2.1. *A* regular sequence $x = \{x_n\}_{n>0}$ *is a sequence of rationals such that*

$$|x_n - x_m| \leq \frac{1}{n} + \frac{1}{m}$$

for all $m, n > 0$. The class of all regular sequences is denoted R_c.

The mentioned equivalence relation of the traditional approach is defined by

$$x \sim y \quad \Longleftrightarrow \quad |x_n - y_n| \leq \frac{2}{n} \text{ for all } n > 0.$$

We, however, will associate to each regular sequence x a cut X_x by

$$X_x := \{r \in \mathbb{Q} : (\exists s > r)(\exists n > 0)(\forall m \geq n) s \leq x_m\}.$$

Proposition 2.2.2. *For all $x, y \in R_c$,*

(i) $X_x \in \mathbb{R}$

(ii) $X_x = X_y$ *iff* $x \sim y$.

Definition 2.2.3. *The* Cauchy reals *make up the class* $\mathbb{R}_c = \{X_x : x \in R_c\}$.

The following proposition and its corollary show the relationship between our two constructions of the continuum.

Proposition 2.2.4. *Let $X \in \mathbb{R}$. Then $X \in \mathbb{R}_c$ iff X is countably infinite.*

Corollary 2.2.5. (**ECST**)

(i) $\mathbb{R}_c$ *is a subfield of* $\mathbb{R}$.

(ii) Assuming $\boldsymbol{AC}_\omega$, $\mathbb{R}_c = \mathbb{R}$.

Depending on which axioms we are working with, either structure of reals may be a proper class. The following reveals when the continuum is a set.

Theorem 2.2.6. (**ECST**)

(i) If $\mathbb{N}^\mathbb{N}$ is a set, then so is $\mathbb{R}_c$.

(ii) Assuming Subset Collection, $\mathbb{R}$ is a set.

Corollary 2.2.7. (**CZF**) *$\mathbb{R}_c$ and $\mathbb{R}$ are sets.*

3 Foundations

The two standard, and classically equivalent, conditions for a relation $<_A$ to be a well-founded relation on a set A are:

(i) Every nonempty subset of A has a $<_A$-least element.

(ii) There is no infinite descending $<_A$ sequence.

In constructive set theory, however, the first is too strong, while the second too weak. Since the importance of well-founded relations is that they allow for proofs by induction and definitions by recursion, we define them directly with this purpose in mind. We then see that the desired properties hold.

3.1 Well-founded relations

We first define our constructive notion of well-foundedness.

Definition 3.1.1. *A subset $X \subseteq A$ is $<_A$-*inductive *if*

$$\forall u \in A[(\forall v \in A)(v <_A u \to v \in X) \to u \in X].$$

$<_A$ is well-founded *if each $<_A$-inductive subset of A equals A.*

Lemma 3.1.2. (**ECST**) *If $<_A$ is a well-founded relation on a set A, then there is no infinite descending $<_A$-sequence.*

Proof (sketch). Suppose, toward contradiction, that there is a function $f : \mathbb{N} \to \mathbb{N}$ such that $f(n+1) < f(n)$ for all n. Then $B := A \backslash f[\mathbb{N}]$ is inductive, and so $B = A$. But $f(0) \notin B$, a contradiction. □

3.2 Induction and recursion

The scheme of Δ_0-induction on ω,

$$\varphi(0) \wedge \forall n \in \omega[\varphi(n) \to \varphi(n+1)] \to (\forall n \in \omega)\varphi(n)$$

where $\varphi(x) \in \Delta_0$, is provable in **ECST**. We show here that by assuming Set Induction, we get induction on ω for arbitrary formulas $\varphi(x)$.

Lemma 3.2.1. **ECST** $+$ Set induction $\vdash$ $\mathbf{IND}_\omega$.

Proof. Assume $\varphi(0) \wedge \forall n \in \omega[\varphi(n) \to \varphi(n+1)]$. We define $\theta(x) :\equiv x \in \omega \to \varphi(x)$ in order to apply Set Induction. So suppose $\forall x \in a\theta(x)$. In order to show $\theta(a)$, let $a \in \omega$. Then $a = 0$ or $a = n+1$ (see [1, Chapter 5] for facts about the natural numbers). If the former, then $\varphi(a)$, and so $\theta(a)$. If the latter, then $n \in a$ implies $\theta(n)$, and so $\varphi(n)$. This gives $\varphi(n+1)$, by our original assumption, which gives $\theta(a)$. So $\forall a[\forall x \in a\theta(x) \to \theta(a)]$, and so, by Set Induction, $\forall a\theta(a)$, which is what we want: $\forall n \in \omega\varphi(n)$. □

Another essential tool in set theory that requires Set Induction is definition by recursion.

Theorem 3.2.2. (**ECST** + Set Induction) *If G is a total $(n+2)$-ary class function, i.e.*

$$\forall \bar{x}yz\exists! uG(\bar{x}, y, z) = u,$$

then there is a total $(n+1)$-ary class function F such that

$$\forall \bar{x}y[F(\bar{x}, y) = G(\bar{x}, y, F\restriction_y)].$$

Proof (sketch). Let

$$\Phi(f, \bar{x}) :\equiv [f \text{ is a function}] \wedge [\text{dom}(f) \text{ is transitive}] \wedge [\forall y \in \text{dom}(f)(f(y) = G(\bar{x}, y, f \restriction y))]$$

and

$$\psi(\bar{x}, y, f) :\equiv [\Phi(f, \bar{x}) \wedge y \in \text{dom}(f)].$$

Then $\forall \bar{x}y\exists! f\psi(\bar{x}, y, f)$ by Set Induction on y. We can then define

$$F(\bar{x}, y) = w \quad :\equiv \quad \exists f[\psi(\bar{x}, y, f) \wedge f(y) = w].$$

□

3.3 Ordinals

With defintion by recursion in hand, we can move on to ordinals.

Definition 3.3.1. *An* ordinal *is a transitive set of transitive sets.*

Lemma 3.3.2. *For a set x, let $x + 1 = x \cup \{x\}$.*

(i) $\alpha + 1 \in \mathbf{ON}$.

(ii) If X is a set of ordinals, then $\bigcup X \in \mathbf{ON}$.

The proof of the following is essentially the same as that of Theorem 3.2.2.

Proposition 3.3.3. (**ECST** + Set Induction) *If G is a total $(n+2)$-ary class function on $V^n \times \mathbf{ON} \times V$, i.e.*

$$\forall \bar{x}\alpha z\exists! uG(\bar{x}, \alpha, z) = u,$$

then there is an $(n+1)$-ary class function $F : V^n \times \mathbf{ON} \to V$ such that

$$\forall \bar{x}\alpha[F(\bar{x}, \alpha) = G(\bar{x}, \alpha, F\restriction_\alpha)].$$

Using Theorem 3.2.2, we can make the following definition.

Definition 3.3.4. (**ECST** + Set Induction) *For any set x,* $\text{rank}(x) := \bigcup\{\text{rank}(y) + 1 : y \in x\}$.

Proposition 3.3.5. (**ECST** + Set Induction)

(i) $\forall x \ \mathrm{rank}(x) \in \mathbf{ON}$.

(ii) $\forall \alpha \ \mathrm{rank}(\alpha) = \alpha$.

Proof (sketch). (i) is proved by Set Induction on x and uses Lemma 3.3.2. (ii) uses induction on α. □

The linearity of the ordinals, which we will look at in §5.2, cannot be proved by our axioms. If we were to force this by requiring that

$$\forall \beta \in \alpha \forall \gamma \in \alpha (\beta \in \gamma \vee \beta = \gamma \vee \gamma \in \beta)$$

for all ordinals α, then Lemma 3.3.2 and Proposition 3.3.5 would fail.

4 Choice

In this section we discuss some principles of choice. We first show that the Axiom of Choice is constructively unacceptable. We then state some constructive choice principles and their interconnections.

First, we define two other principles that will be involved. Restricted Excluded Middle, **REM**, is the schema $\varphi \vee \neg\varphi$ for all $\varphi \in \Delta_0$. **EM** is the same but for all formulas φ. Exponentiation states that for any sets a and b, the class ${}^a b$ of all functions from a to b is a set.

4.1 AC

Proposition 4.1.1. **ECST** + Exponentiation + **REM** $\vdash$ Powerset.

Proof. Let $u \subseteq 1$. By **REM**, $0 \in u \vee 0 \notin u$. So $u = 1 \vee u = 0$, giving $u \in 2$. Thus $\mathcal{P}(1) \subseteq 2$. Then $\mathcal{P}(1) = \{u \in 2 : u \subseteq 1\}$ is a set by Δ_0-Separation.

Now let a be any set, and define $b := {}^a(\mathcal{P}(1))$, which is a set by Exponentiation. Then

$$c := \{\{x \in a : g(x) = 1\} : g \in b\}$$

is a set by Replacement. Note that if $y \in c$ then $y \subseteq a$. But also, if $y \subseteq a$, then

$$y = \{x \in a : \chi_y(x) = 1\}$$

(where $\chi_y(x) := \{w \in 1 : x \in y\}$ is an element of b), and so $y \in c$. Thus $\mathcal{P}(a) = \{y \in c : y \subseteq a\} = c$ is a set. □

In fact, the strength of **ECST** + Exponentiation + **REM** exceeds that of classical type theory with extensionality.

The Axiom of Choice, **AC**, states that for all sets A and functions F with domain A, if

$$\forall i \in A \exists y \in F(i),$$

then there is a function f with domain A such that

$$\forall i \in A f(i) \in F(i).$$

Proposition 4.1.2.

(i) **ECST** + Exponentiation + Separation + **AC** = **ZFC**.

(ii) **ECST** + **AC** $\vdash$ **REM**.

(iii) **ECST** + Exponentiation + **AC** $\vdash$ Powerset.

Proof. (i) Let φ be any formula. Define

$$X = \{n \in \omega : n = 0 \vee [n = 1 \wedge \varphi]\},$$

$$Y = \{n \in \omega : n = 1 \vee [n = 0 \wedge \varphi]\},$$

which are sets by full Separation. We have $\forall z \in \{X, Y\} \exists k \in \omega(k \in z)$. By **AC**, we get a choice function f on $\{X, Y\}$ such that

$$\forall z \in \{X, Y\}[f(z) \in \omega \wedge f(z) \in z].$$

So $f(X) \in X$ and $f(Y) \in Y$. Since $\forall m, n \in \omega(n = m \vee n \neq m)$,

$$f(X) = f(Y) \vee f(X) \neq f(Y).$$

If $f(X) = f(Y)$, then φ, by the definitions of X and Y. If $f(X) \neq f(Y)$, then $X \neq Y$, and so $\neg\varphi$. Thus $\varphi \vee \neg\varphi$. So, assuming **AC** and Separation, we got **EM**. This, together with

$$\textbf{ECST} + \text{Exponentiation} + \textbf{EM} = \textbf{ZF},$$

proves (i).

(ii) If $\varphi \in \Delta_0$, then X and Y are sets by Δ_0-Separation. The rest of the previous proof gives **REM**.

(iii) This follows from (ii) and Proposition 4.1.1.

□

4.2 Constructive Choice Principles

Now that we have seen some unwelcomed consequences of full **AC**, we look at several weaker versions.

The Axiom of Countable Choice, $\mathbf{AC}_\omega$, states that if F is a function with domain ω such that

$$\forall i \in \omega \exists y \in F(i),$$

then there is a function f with domain ω such that

$$\forall i \in \omega f(i) \in F(i).$$

The Dependent Choices Axiom, **DC**, states that if a is a set and $R \subseteq a \times a$ such that

$$(\forall x \in a)(\exists y \in a)xRy$$

and $b_0 \in a$, then there is a function $f : \omega \to a$ such that $f(0) = b_0$ and

$$(\forall n \in \omega)f(n)Rf(n+1).$$

The Relativized Dependent Choices Axiom, **RDC**, states that for any formulas φ and ψ, if

$$\forall x[\varphi(x) \to \exists y(\varphi(y) \land \psi(x,y))]$$

and $\varphi(b_0)$, then there is a function f with domain ω such that $f(0) = b_0$ and

$$(\forall n \in \omega)[\varphi(f(n)) \land \psi(f(n), f(n+1))].$$

The Bounded Relativized Dependent Choices Axiom, **bRDC**, states that for any Δ_0-formulas φ and ψ, if

$$\forall x \in a[\varphi(x) \to \exists y \in a(\varphi(y) \land \psi(x,y))]$$

and $b_0 \in a \land \varphi(b_0)$, then there is a function $f : \omega \to a$ such that $f(0) = b_0$ and

$$(\forall n \in \omega)[\varphi(f(n)) \land \psi(f(n), f(n+1))].$$

Proposition 4.2.1. (**ECST**) **RDC** $\Longrightarrow$ **bRDC** $\Longleftrightarrow$ **DC** $\Longrightarrow$ $\mathbf{AC}_\omega$.

Proof. That **bRDC** and **RDC** imply **DC** is trivial. That **DC** implies **bRDC** relies on a result we've omitted. So we just prove that **DC** implies $\mathbf{AC}_\omega$.

If $z = \langle x, y\rangle$, let $1^{st}(z) := x$ and $2^{nd}(z) := y$.

Let F be a function with domain ω such that $\forall i \in \omega \exists x \in F(i)$. Define

$$A := \{\langle i, u\rangle : i \in \omega \land u \in F(i)\}$$

(which is a set by Union, Cartesian Product, and Δ_0-Separation) and

$$R := \{\langle x, y\rangle \in A \times A : 1^{st}(x) + 1 = 1^{st}(y)\},$$

and let $a_0 = \langle 0, x_0\rangle$ for some $x_0 \in F(0)$. Then $\forall x \in A \exists y \in A(xRy)$, so, by **DC**, there is a function $g : \omega \to A$ such that $g(0) = a_0$ and

$$\forall i \in \omega[1^{st}(g(i+1)) = 1^{st}(g(i)) + 1].$$

Defining a function f on ω by $f(i) := 2^{nd}(g(i))$, we get $\forall i \in \omega f(i) \in F(i)$. □

5 Principles to avoid

In this section we look at two major principles which are true in **ZF**: The Foundation Schema/Axiom and the Linearity of Ordinals. It is shown that these are too powerful for constructive set theory.

5.1 Foundation

The Foundation Shema is

$$\exists x \varphi(x) \to \exists x[\varphi(x) \land \forall y \in x \neg\varphi(y)]$$

for all formulas φ. The Foundation Axiom is

$$\forall x[\exists y(y \in x) \to \exists y(y \in x \land \forall z \in y(z \notin z))].$$

Proposition 5.1.1.

(i) **CZF** + Foundation Schema = **ZF**.

(ii) **CZF** + Separation + Foundation Axiom = **ZF**.

(iii) **CZF** + Foundation Axiom $\vdash$ **REM**.

(iv) **CZF** + Foundation Axiom $\vdash$ Powerset.

Proof. (i) Let φ be any formula. We show $\varphi \vee \neg\varphi$. Let $S_\varphi := \{x \in \omega : x = 1 \vee [x = 0 \wedge \varphi]\}$. (Note that S_φ need not be a set.) $1 \in S_\varphi$. By the Foundation Schema, there is an $x_o \in S_\varphi$ such that $\forall y \in x_0 y \notin S_\varphi$. Then, by the definition of S_φ, $x_0 = 1 \vee [x_0 = 0 \wedge \varphi]$. If $x_0 = 1$, then $0 \notin S_\varphi$, so $\neg\varphi$. If $x_0 = 0 \wedge \varphi$, then φ. So **EM** holds.

(ii) Assuming Separation, S_φ is a set. Then the previous proof goes through with only the Foundation Axiom.

(iii) If $\varphi \in \Delta_0$, then S_φ is a set by Δ_0-separation. So the proof of (ii) gives **REM**.

(iv) By (iii) and the fact that **CZF** proves Exponentiation, Powerset follows from Proposition 4.1.1.

□

In fact, the strength of **CZF** + Foundation Axiom exceeds that of classical type theory with extensionality.

5.2 Linearity

At the end of §3, we noted that requiring ordinals to be linear has negative, though minor, consequences. Here we show that the consequences are worse than foreshadowed. Linearity of Ordinals states that

$$\forall \alpha\beta \in \mathbf{ON}[\alpha \in \beta \vee \alpha = \beta \vee \beta \in \alpha].$$

Proposition 5.2.1.

(i) **CZF** + Linearity of Ordinals $\vdash$ Powerset.

(ii) **CZF** + Linearity of Ordinals $\vdash$ **REM**.

(iii) **CZF** + Linearity of Ordinals + Separation = **ZF**.

Proof. (i) 1 and 2 are ordinals. If $u \subseteq 1$, then $\forall z \in u(z = 0)$, so u is an ordinal. By linearity of ordinals,

$$\forall u \subseteq 1[u \in 2 \vee u = 2 \vee 2 \in u].$$

The last two disjuncts cannot be true, so $\forall u \subseteq 1[u \in 2]$. Thus $\mathcal{P}(1) = \{u \in 2 : u \subseteq 1\}$ is a set. From here, just as in the proof of Proposition 4.1.1, we get Powerset.

(ii) Let $\varphi \in \Delta_0$. Then $\alpha := \{n \in \omega : n = 0 \wedge \varphi\}$ is a set by Δ_0-Separation, and in fact an ordinal since it's a subset of 1. So, by Linearity of Ordinals, $\alpha \in 1 \vee \alpha = 1 \vee 1 \in \alpha$. If $\alpha \in 1$, then $\alpha = 0$, giving $\neg\varphi$. If $\alpha = 1$, then φ. (And $1 \notin \alpha$.) Therefore, $\varphi \vee \neg\varphi$, giving **REM**.

(iii) If φ is any formula, then $\alpha := \{n \in \omega : n = 0 \wedge \varphi\}$ is a set by Separation. Then the previous proof gives $\varphi \vee \neg\varphi$, giving **EM**.

□

References

[1] P. Aczel and M. Rathjen. *Notes on Constructive Set Theory*. Draft, 2005.